

Adjustment under gravity in a rotating channel

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What transient motions occur as a fluid responds to gravitational forces in a rotating channel, and what equilibrium does the fluid adjust to? This problem is studied to illustrate how boundaries affect the process of adjustment to a geostrophic equilibrium. In particular, linear solutions are found for an infinitely long channel of constant width and constant depth when there is an initial discontinuity in the level of the free surface. The results are summarized in the figures, and can be described in terms of Poincaré waves and Kelvin waves. When the channel is wide compared with the Rossby radius, the final state involves a current of that width which follows the left-hand boundary (for Northern-Hemisphere rotation) to the position of the initial discontinuity, then crosses the channel and continues downstream along the right-hand wall.

1. Introduction

Much of the behaviour of the atmosphere and ocean can be understood in terms of a competition between various forcing effects and gravity. Gravity can be dubbed the restoring force which tries to bring the system towards an equilibrium state; the competing forces can be thought of as agents which tend to disrupt such an equilibrium. A major step towards understanding the complex situations which arise in practice is gained by studying how systems behave when the 'disruptive' forces are absent and gravity is the force which causes the system to change. This paper explores one such situation.

Historically, geophysical applications were amongst the first to be tried for the equations of fluid motion after these were developed by Euler (1755). In particular, Laplace (1778/9) derived what are now called the 'Laplace tidal equations', which describe motion under gravity on a rotating sphere in the presence of tide-generating forces, i.e. those due to the gravitational attraction of the sun, moon, etc. He began the study of wave motions under gravity by finding (in a section called "On Waves" in the third part of his paper) the dispersion relation for gravity waves in the absence of rotation. Kelvin (1879) found the wave solutions for a rotating channel: not only the edge wave named after him, but also the so-called 'Poincaré wave', which was further discussed in a book on tidal theory by Poincaré (1910).

The problem considered in this paper is of a type studied by Rossby (1937, 1938) in order to clarify the mechanism by which pressure and velocity distributions in the atmosphere and ocean tend to undergo mutual adjustment, that is,

approach a 'geostrophic' equilibrium wherein pressure gradients are balanced by the Coriolis acceleration. The method is simply to consider an initial-value problem where this balance is not satisfied at the beginning, and thereby study how quickly and by what means the adjustment to equilibrium takes place. Rossby himself did not calculate full details of the adjustment process, but found some integrals of the motion. In particular, he was able to find the final state without knowing details of the adjustment process. This remarkable result used the principle which is known as 'conservation of potential vorticity'. The relevant equation in fact appeared in Kelvin's paper of 1879, but does not lead to such a noteworthy result when the initial perturbation is zero. Rossby also drew attention to the fundamental length scale of the adjustment problem, which he called the 'radius of deformation'. An excellent account of subsequent work on Rossby's problem is given by Blumen (1972).

In the particular case considered by Rossby, there was assumed to be no dependence on one of the horizontal co-ordinates. The aim of this paper is to calculate the effects of boundaries for the case where the fluid is in a channel of finite width. As will be seen, this introduces some fundamentally new features into the problem because the Kelvin wave associated with each boundary can propagate information in only one direction.

The motivation for studying this type of problem mainly comes from considerations of internal adjustment in the ocean or in other large bodies of water like the Great Lakes, for effects of coastal boundaries are important in such cases. Examples where the channel geometry might be of particular interest are straits like the Bosphorus or the Straits of Gibraltar, where there are density differences between the water masses at each end, or in deep channels connecting ocean basins.

2. Definition of the problem

The problem to be considered is the adjustment under gravity of a stratified fluid in a channel of constant width and constant depth. The perturbations from the rest state, where pressure and density are functions only of depth, are assumed to be infinitesimal so that linear analysis can be applied. Also the perturbations are assumed to have horizontal scales large compared with the vertical scales so that the 'long-wave' or 'shallow-water' or, equivalently, the 'hydrostatic' approximation can be used. In these circumstances, the motion can be separated into normal modes (see Krauss 1966; Gill & Clarke 1974). For each normal mode, the variations in the horizontal and in time are the same as for a homogeneous fluid with an appropriate 'equivalent depth'. For simplicity, the initial perturbation will be assumed to have energy in only one of the normal modes. The problem is therefore equivalent to one of adjustment of a homogeneous fluid of appropriate equilibrium depth H , so notation appropriate to the equivalent homogeneous problem will be used.

Let (x, y, z) be rectangular co-ordinates such that the z axis points upwards and the sides of the channel are parallel to the x axis. The axes are fixed in a frame that rotates uniformly about a vertical axis with angular velocity $\frac{1}{2}f$. Thus

f is the Coriolis parameter. Let η be the upward displacement of the free surface from its equilibrium position and g be the gravitational acceleration. If the above-mentioned assumption is made (the shallow-water or hydrostatic approximation) the horizontal velocity components (u, v) corresponding to the coordinates (x, y) are independent of depth. If t represents time, the governing equations are the momentum equations

$$\partial u / \partial t - fv = -g \partial \eta / \partial x, \quad (2.1)$$

$$\partial v / \partial t + fu = -g \partial \eta / \partial y \quad (2.2)$$

and the continuity equation

$$\frac{\partial \eta}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \quad (2.3)$$

The system can be made *non-dimensional* by choosing f^{-1} as the unit of time and the Rossby radius of deformation $(gH)^{1/2}/f$ as the unit of horizontal distance. The unit of vertical displacement can be set by the initial distribution, and the unit of horizontal velocity can be set at $(g/H)^{1/2}$ times this value. This choice of units is equivalent to putting $f = g = H = 1$ in the equations. The origin for y can be chosen such that the channel walls are at $y = \pm \frac{1}{2}b$, i.e. the channel width is b times the radius of deformation.

The problem is to solve (2.1)–(2.3) for some initial distribution

$$\eta = \eta_0, \quad u = u_0, \quad v = v_0 \quad \text{at} \quad t = 0, \quad (2.4)$$

where $\nabla \eta_0$, u_0 and v_0 are assumed to be non-zero only in a finite region. The boundary conditions are that

$$v = 0 \quad \text{at} \quad y = \pm \frac{1}{2}b, \quad (2.5)$$

and that energy cannot propagate in from $x = \pm \infty$. For simplicity attention will be mainly devoted to the particular case where

$$u_0 = 0, \quad v_0 = 0, \quad \eta_0 = \text{sgn } x, \quad (2.6)$$

with $\text{sgn } x$ defined as the step function

$$\text{sgn } x = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0. \end{cases} \quad (2.7)$$

The function (2.7) is to be interpreted as a jump which is narrow on the unit scale (i.e. the radius of deformation) but wide compared with the depth of fluid. This is necessary for the shallow-water approximation to be consistent. Clearly the radius of deformation must be large compared with the depth for the whole approach to be valid in the first place. Methods used for solving the particular case with initial condition (2.6) may be readily generalized.

Before calculating the solution for finite channel width, two limiting cases where the solution is well known will be discussed. The first limit is that of a narrow channel. If the channel is narrow enough, motion normal to the channel walls is suppressed and with it the associated Coriolis acceleration. Thus the

motion is the same as in a non-rotating channel. The second limit is that where the channel width is infinite. In this limit, the problem is the classical 'Rossby adjustment problem', where there is no dependence on one of the spatial coordinates.

3. The narrow-channel limit

Consider the limit $b \rightarrow 0$ for the case where the initial profiles u_0, η_0 and v_0 are independent of y . In this limit, the walls are so close together that motion normal to them is suppressed, i.e. $v = 0$. Thus (2.1), using the special units of length and time, becomes

$$\partial u / \partial t = -\partial \eta / \partial x \quad (3.1)$$

and (2.3) becomes

$$\partial \eta / \partial t + \partial u / \partial x = 0. \quad (3.2)$$

These are also the equations for a *non-rotating* system. If u is eliminated, the wave equation results:

$$\partial^2 \eta / \partial t^2 = \partial^2 \eta / \partial x^2. \quad (3.3)$$

The solution

$$\eta = \mathcal{N}(x, t), \quad u = \mathcal{U}(x, t) \quad (3.4)$$

of this system is well known. In the special case where the initial state is (2.6), \mathcal{N} and \mathcal{U} are given by

$$\left. \begin{aligned} \mathcal{N}(x, t) &= \frac{1}{2}[\operatorname{sgn}(x+t) + \operatorname{sgn}(x-t)], \\ \mathcal{U}(x, t) &= \frac{1}{2}[-\operatorname{sgn}(x+t) + \operatorname{sgn}(x-t)]. \end{aligned} \right\} \quad (3.5)$$

This solution consists of two steps (representing wave fronts) which move in either direction from the initial discontinuity with unit speed (or, in dimensional terms, with speed $(gH)^{1/2}$). Ahead of the wave fronts, the solution is no different from the initial state. Behind the wave fronts, the solution is

$$\eta = 0, \quad u = -1, \quad (3.6)$$

i.e. the surface displacement is zero and a current flows from the region of high surface elevation towards the region of low surface elevation.

The above solution is the first approximation for the narrow-channel limit, and higher approximations can be obtained by expanding u , v and η as power series in b . Equation (2.2), for example, shows there is a finite cross-channel slope because of rotation effects, but the *difference* in elevation between the two sides of the channel is small, and so this gives a contribution to η of order b . In the formal procedure, x variations are considered to be of order unity as $b \rightarrow 0$ while y has the scale b . The highest-order term in the expansion for v is then of order b and each term in the expansion can be split into two parts: one for which the time scale is of order unity and one for which the time scale is of order b .

Given the above information, the expansion procedure is straightforward, so details will not be given. However, it may be of interest to point out that this procedure can be used in another context to compute effects of rotation on seiches and tides in gulfs and estuaries whose width is not large compared with the radius of deformation (cf. Taylor 1921; Hendershott & Speranza 1971). To

first order, the non-rotating models are applicable and effects of variable cross-section can be incorporated if desired. Expansion in the ratio of the mean width to the radius of deformation gives the rotation effects, as in the above problem.

4. Equations derived from the governing equations

Consider first the *energy* equation, which can be derived from the governing equations by multiplying (2.1), (2.2) and (2.3) by u , v and η respectively and adding. Using the special units of length and time, this gives

$$\frac{1}{2} \frac{\partial}{\partial t} (u^2 + v^2 + \eta^2) + \frac{\partial}{\partial x} (u\eta) + \frac{\partial}{\partial y} (v\eta) = 0. \quad (4.1)$$

Integrating over a finite section of the channel $|x| \leq L$, $|y| \leq \frac{1}{2}b$ and with respect to time gives

$$P_f - P_i + K_f - K_i = R, \quad (4.2)$$

where

$$P = \int_{y=-\frac{1}{2}b}^{\frac{1}{2}b} \int_{x=-L}^L \frac{1}{2} \eta^2 dx dy \quad (4.3)$$

is the potential energy,

$$K = \int_{y=-\frac{1}{2}b}^{\frac{1}{2}b} \int_{x=-L}^L \frac{1}{2} (u^2 + v^2) dx dy \quad (4.4)$$

is the kinetic energy, the suffixes i and f refer to initial and final values respectively, and

$$R = \int_0^\infty \int_{y=-\frac{1}{2}b}^{\frac{1}{2}b} ((u\eta)_L - (u\eta)_{-L}) dy dt \quad (4.5)$$

represents the energy lost by radiation, i.e. by waves carrying energy out of the region concerned. In practice, L can be chosen such that this region includes the whole of the finite region where $\nabla\eta_0$, u_0 and v_0 are non-zero.

For example, the energy changes for the narrow-channel, or non-rotating, case considered in § 3 are as follows:

$$P_i = bL, \quad P_f = 0, \quad K_i = 0, \quad K_f = bL, \quad R = 0. \quad (4.6)$$

In other words the initial energy, which is all potential energy in the example considered, is all converted into kinetic energy. A look at time dependence shows that all this conversion takes place in the time L it takes a long gravity wave to cross half the region concerned.

The second derived equation to be considered is the *potential-vorticity* equation, which was used to good effect by Rossby. Its linear form (using the special units of length and time)

$$\frac{\partial}{\partial t} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta \right) = 0 \quad (4.7)$$

is obtained by subtracting (2.3) and the y derivative of (2.1) from the x derivative of (2.2). The importance of this equation lies in the fact that it can be integrated immediately, so giving a connexion between the initial and the final state of the system. For the linear case, the integrated form is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \eta_0. \quad (4.8)$$

Equations in a single variable can be obtained by using (2.1) and (2.2) to eliminate any two of the three variables. For instance, the equation for η is

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} - \frac{\partial^2 \eta}{\partial t^2} - \eta = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \eta_0, \quad (4.9)$$

the equation for v is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial t^2} - v = \frac{\partial}{\partial x} \left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \eta_0 \right) \quad (4.10)$$

and the equation for u is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} - u = -\frac{\partial}{\partial y} \left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \eta_0 \right). \quad (4.11)$$

5. The infinitely wide channel

If the initial state is independent of y , so is the solution at all subsequent times, so equations (2.1)–(2.3) in the special co-ordinate system become

$$\partial u / \partial t - v = -\partial \eta / \partial x, \quad (5.1)$$

$$\partial v / \partial t + u = 0, \quad (5.2)$$

$$\partial \eta / \partial t + \partial u / \partial x = 0. \quad (5.3)$$

The potential-vorticity equation for this case is obtained by taking the x derivative of (5.2) and subtracting (5.3). Then, by integrating with respect to time, the special case of (4.4) obtained is

$$\partial v / \partial x - \eta = \partial v_0 / \partial x - \eta_0. \quad (5.4)$$

The equations satisfied by u , v and η are (4.9), (4.10) and (4.11) with the y derivatives zero. Studies of this problem are reviewed by Blumen (1972).

Rossby (1938) showed the power of the potential-vorticity concept by finding the final state without considering changes with time. Since the time derivative is zero, (5.1) gives v in terms of η and substitution in (5.4) gives

$$\partial^2 \eta / \partial x^2 - \eta = \partial v_0 / \partial x - \eta_0, \quad (5.5)$$

a special case of (4.9). This equation may be solved by elementary methods. Rossby's solution was for an initial state where $v_0 = \eta_0 = 0$ and u_0 is non-zero (and constant) only in a finite strip. The solution for the initial state (2.6) is

$$u = 0, \quad \eta = M(x), \quad v = M'(x), \quad (5.6)$$

where

$$M(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ -1 + e^{-x}, & x \leq 0, \end{cases} \quad (5.7)$$

and is shown in figure 1.

The implications of this result in terms of energy are of fundamental importance. In the narrow-channel or non-rotating case considered in §3, *all* the

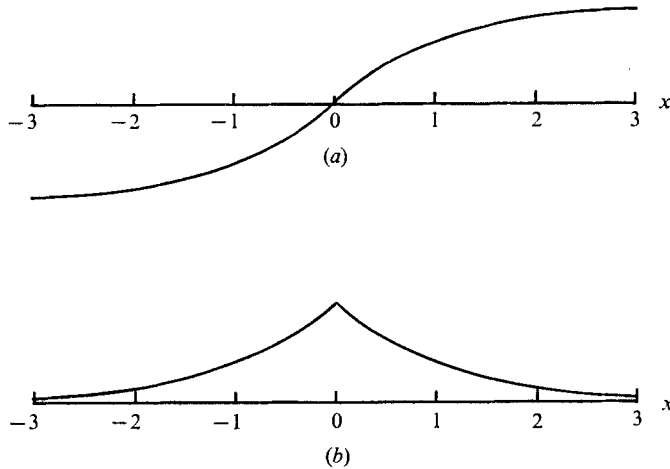


FIGURE 1. The steady solutions (a) $\eta = M(x)$ and (b) $v = M'(x)$ for the wide-channel case. The unit of length is the Rossby radius.

potential energy initially present is ultimately converted into kinetic energy. In the rotating system without walls, this is no longer true. Only a finite portion of the initial potential energy is 'released', i.e. converted into kinetic energy or used up in some other way. The amount released can be calculated using (5.7). By the definition (4.3), the difference between the final and initial potential energies is, per unit length in the y direction,

$$\frac{1}{2} \int_{-\infty}^{\infty} (\text{sgn}(x)^2 - M(x)^2) dx = 1\frac{1}{2}. \quad (5.8)$$

On the other hand, the kinetic energy of the final state is only

$$\frac{1}{2} \int_{-\infty}^{\infty} M'(x)^2 dx = \frac{1}{2}. \quad (5.9)$$

What happens to the remaining unit of energy?

By (4.2), there are two possibilities. Either the energy is lost by radiation or the solution in any finite region never reaches a steady state. It turns out that the former alternative is the correct one, the energy being radiated by Poincaré waves. These have u , v and η proportional to $\exp(ikx - i\omega t)$ and are free solutions of (4.9)–(4.11). By substitution in these equations, the dispersion relation is

$$\omega^2 = 1 + k^2. \quad (5.10)$$

The short waves are little affected by rotation and move at almost unit speed like ordinary gravity waves in a non-rotating system. The long waves, on the other hand, all have frequencies close to the inertial frequency, which is unity in the co-ordinate system being used.

The fact that the solution approaches the steady one in any finite region, and does not simply oscillate about it with undiminishing amplitude, was discovered by Cahn (1945), who solved the initial-value problem with Rossby's initial values.

The approach outlined below is somewhat different (*a*) because of the representation in terms of Poincaré waves and (*b*) since certain definitions and properties are mentioned because of their usefulness later in discussing the finite-width channel. In particular, let

$$v = V(x, t) \quad (5.11)$$

be the solution of the transient problem for the given initial conditions. Then the solutions

$$\eta = N(x, t), \quad u = U(x, t) \quad (5.12)$$

for η and u are given explicitly in terms of V by (5.2) and (5.4), that is

$$\left. \begin{aligned} u = U(x, t) &\equiv -\partial V(x, t)/\partial t, \\ \eta = N(x, t) &\equiv \partial V(x, t)/\partial x - dv_0(x)/dx + \eta_0(x). \end{aligned} \right\} \quad (5.13)$$

Thus there is particular value in looking first for the solution for V , which satisfies (4.10), i.e.

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t^2} - V = \frac{d}{dx} \left(\frac{dv_0}{dx} - \eta_0 \right). \quad (5.14)$$

The solution V of (5.14) can be written as the sum of a particular solution and an appropriate combination of Poincaré waves which are solutions of the homogeneous equation. For the particular initial condition (2.6), a suitable special solution is the steady one, $V = M'(x)$. By symmetry, Poincaré waves will only appear in the combination

$$\cos(kx + \omega t) + \cos(kx - \omega t) = 2 \cos kx \cos \omega t,$$

so V can be written in the form

$$V(x, t) = M'(x) + \frac{2}{\pi} \int_0^\infty a(k) \cos \{t(1+k^2)^{\frac{1}{2}}\} \cos kx \, dk, \quad (5.15)$$

where the dispersion relation (5.10) has been used to give ω in terms of k . The initial condition requires

$$M'(x) + \frac{2}{\pi} \int_0^\infty a(k) \cos kx \, dk. \quad (5.16)$$

The function $a(k)$ can be found by taking the appropriate Fourier integral, or simply by looking up tables. Erdélyi *et al.* (1954, p. 8) give the result, when M is given by (5.7), that

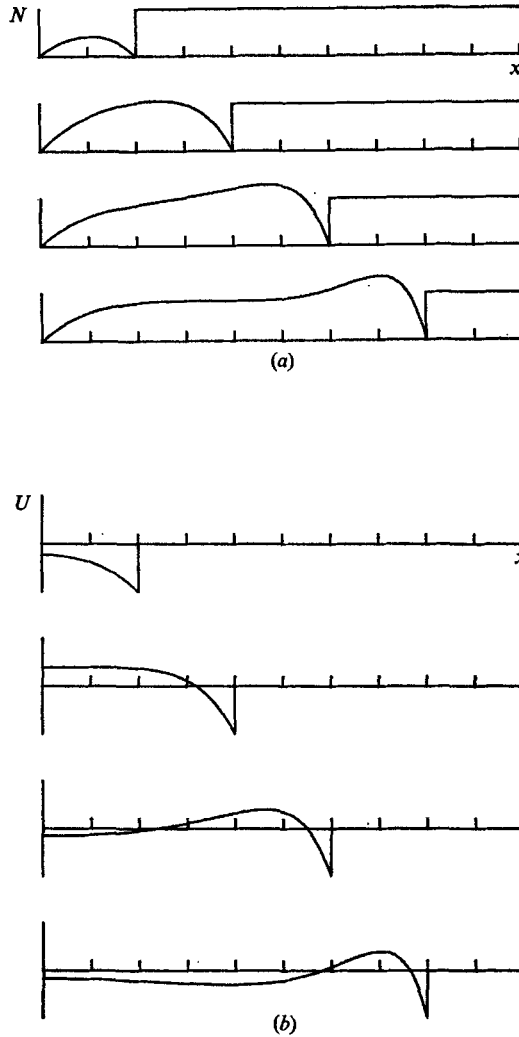
$$a(k) = -1/(1+k^2). \quad (5.17)$$

The expression for U can be obtained from (5.12):

$$U(x, t) = -\frac{2}{\pi} \int_0^\infty (1+k^2)^{-\frac{1}{2}} \sin \{t(1+k^2)^{\frac{1}{2}}\} \cos kx \, dk \quad (5.18)$$

$$= \begin{cases} -J_0\{t^2 - x^2\}^{\frac{1}{2}} & \text{for } |x| < t, \\ 0 & \text{for } |x| > t, \end{cases} \quad (5.19)$$

by Erdélyi *et al.* (1954, p. 26; see also Dickinson 1969). This solution is given in Morse & Feshbach (1953, p. 139) and illustrated in figure 2. Morse & Feshbach call (4.11), with zero y derivatives, the Klein-Gordon equation, which can be derived for a stretched string embedded in an elastic medium. Equation (5.19)



FIGURES (a, b). For legend see page 612.

is then the solution for a point impulse. Because the short waves ($k \gg 1$) behave as in the non-rotating case, a wave front moves out at unit speed just as in the non-rotating case. The group velocity

$$d\omega/dk = k/(1 + k^2) \tag{5.20}$$

cannot exceed unity. Dispersion affects all waves other than very short ones, resulting in a 'wake' being left behind after the wave front has passed. Since $t^2 - x^2 = (t - x)(t + x)$, the length scale of the solution just behind the front $x = t$ decreases in inverse proportion to the time.

There are various ways of computing solutions to the problem. The one used to compute the solutions shown in figure 2 is based on the expression (5.19) for u . From this, v can be found by integrating (5.2) and η by integrating (5.3) to

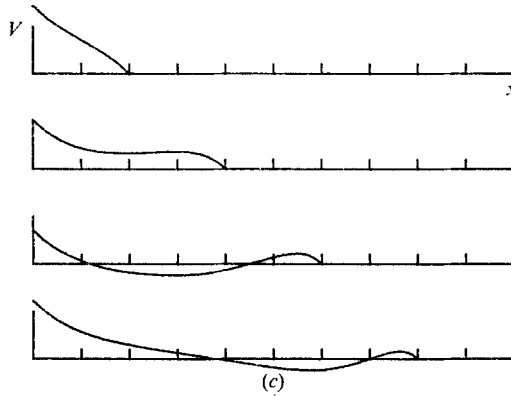


FIGURE 2. The y -independent solutions for the infinitely wide channel for (a) the surface elevation $N(x, t)$, (b) the velocity component $U(x, t)$ perpendicular to the initial surface discontinuity and (c) the velocity component $V(x, t)$ parallel to the initial surface discontinuity. The solutions are only shown for the region $x > 0$ as the solutions elsewhere can be found from the symmetry properties. Solutions are shown two units of time apart, starting at $t = 2$. (The unit of time is f^{-1} and the unit distance is the Rossby radius.) Disturbances move out at unit speed from the initial discontinuity trailing a 'wake' of Poincaré waves behind them. Eventually the solution approaches that shown in figure 1.

obtain an integral expression with a Bessel-function kernel. Such a representation can be used for any initial conditions, and Cahn (1945) used this form for Rossby's initial conditions. For the case (2.6), the result is

$$v = V(x, t) = \begin{cases} \int_0^{(t^2-x^2)^{\frac{1}{2}}} (x^2+r^2)^{-\frac{1}{2}} J_0(r) r dr & \text{for } |x| < t, \\ 0 & \text{for } |x| > t, \end{cases} \quad (5.21)$$

$$\eta = N(x, t) = \begin{cases} x \int_0^{(t^2-x^2)^{\frac{1}{2}}} (x^2+r^2)^{-\frac{1}{2}} J_1(r) dr & \text{for } |x| < t, \\ \text{sgn } x & \text{for } |x| > t. \end{cases} \quad (5.22)$$

These solutions have the undisturbed value for $|x| > t$ and various properties are easily demonstrated, e.g. that $v \rightarrow 1$ at $x = 0$ as $t \rightarrow \infty$. An alternative method is based on fast Fourier transforms and uses (5.15). It is not so accurate in the neighbourhood of the wave front, but is useful for computing wavenumber or frequency spectra of the energy contained in the Poincaré waves.

6. The finite-width channel

The solution for the finite-width channel contains elements of both limiting cases, but also has distinctive features not discernible in either limit. The best starting point is to find the solution for v since the boundary conditions apply to v . The requirement is therefore to solve (4.10) subject to the boundary condition (2.5). For simplicity of presentation, suppose that the initial values u_0 ,

v_0 and η_0 are functions of x only. Because of the boundary conditions (2.5), a Fourier series expansion of the form

$$v = \Sigma \hat{v}_m(x, t) \cos my \quad (6.1)$$

with
$$m = (2n + 1)\pi/b, \quad n = 0, 1, 2, 3, \dots, \quad (6.2)$$

is appropriate. Equations for the \hat{v}_m are obtained by multiplying (4.10) by $(2/b) \cos my$ and integrating with respect to y from $-\frac{1}{2}b$ to $\frac{1}{2}b$. This gives

$$\frac{\partial^2 \hat{v}_m}{\partial x^2} - \frac{\partial^2 \hat{v}_m}{\partial t^2} - (1 + m^2) \hat{v}_m = \alpha_m \frac{d}{dx} \left(\frac{dv_0}{dx} - \eta_0 \right), \quad (6.3)$$

where
$$\alpha_m = (-1)^n 4(mb)^{-1}$$

is the coefficient in the Fourier expansion

$$\Sigma \alpha_m \cos my = \begin{cases} 1, & |y| < \frac{1}{2}b, \\ 0, & |y| = \frac{1}{2}b. \end{cases} \quad (6.4)$$

Note that \hat{v}_m satisfies an equation of the same form as (5.12). It can therefore be solved by the same methods, e.g. as the sum of a steady solution and a Fourier integral in terms of Poincaré waves.

Before calculating solutions of (6.3) for special cases, consider how η and u may be determined once the solution for v is known. One method of doing this is to express η and u as a sum of an odd part and an even part, i.e.

$$\begin{aligned} u^{\text{od}}(x, y, t) &= \frac{1}{2}[u(x, y, t) - u(x, -y, t)], \\ u^{\text{ev}}(x, y, t) &= \frac{1}{2}[u(x, y, t) + u(x, -y, t)]. \end{aligned}$$

The equations relating the variables u^{od} , u^{ev} , η^{od} , η^{ev} and v are the odd and even parts of (2.1), (2.2), (2.3) and (4.8). Only six of these equations are independent, these being

$$u^{\text{od}} = -\partial \eta^{\text{ev}} / \partial y, \quad \eta^{\text{od}} = -\partial u^{\text{ev}} / \partial y, \quad (6.5), (6.6)$$

$$\partial u^{\text{ev}} / \partial t - v = -\partial \eta^{\text{ev}} / \partial x, \quad (6.7)$$

$$\partial v / \partial t + u^{\text{ev}} = -\partial \eta^{\text{od}} / \partial y, \quad (6.8)$$

$$\partial \eta^{\text{ev}} / \partial t + \partial u^{\text{ev}} / \partial x = 0, \quad (6.9)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u^{\text{od}}}{\partial y} - \eta^{\text{ev}} = \frac{dv_0}{dx} - \eta_0. \quad (6.10)$$

Equations (6.5) and (6.6) can be used to eliminate u^{od} and η^{od} , so (6.8) and (6.10) become

$$u^{\text{ev}} - \partial^2 u^{\text{ev}} / \partial y^2 = \partial v / \partial t \quad (6.11)$$

and

$$\eta^{\text{ev}} - \frac{\partial^2 \eta^{\text{ev}}}{\partial y^2} = \frac{\partial v}{\partial x} - \frac{dv_0}{dx}(x) + \eta_0(x). \quad (6.12)$$

Taking note of the Fourier expansions (6.1) and (6.4), these integrate to give

$$u^{\text{ev}} = -\Sigma (1 + m^2)^{-1} (\partial \hat{v}_m / \partial t) \cos my + \mathcal{U}(x, t) \cosh y \operatorname{sech} \frac{1}{2}b, \quad (6.13)$$

$$\eta^{\text{ev}} = \Sigma (1 + m^2)^{-1} \left[\frac{\partial \hat{v}_m}{\partial x} + \alpha_m \left(\eta_0 - \frac{dv_0}{dx} \right) \right] \cos my + \mathcal{N}(x, t) \cosh y \operatorname{sech} \frac{1}{2}b, \quad (6.14)$$

where, for the moment, \mathcal{U} and \mathcal{N} are functions of x and t to be determined. It will turn out, however, that they are the functions defined in § 3.

To find \mathcal{U} and \mathcal{N} , substitute (6.13) and (6.14) in (6.7) and (6.9). These give respectively

$$\partial\mathcal{U}/\partial t = -\partial\mathcal{N}/\partial x, \tag{6.15}$$

$$\partial\mathcal{N}/\partial t + \partial\mathcal{U}/\partial x = 0, \tag{6.16}$$

showing that \mathcal{U} and \mathcal{N} satisfy the same equations as the functions defined in § 3. It remains to show that they satisfy the same initial conditions. First consider (6.13). At $t = 0$, this becomes

$$u_0(x) = A(x, y) + \mathcal{U}(x, 0) \cosh y \operatorname{sech} \frac{1}{2}b, \tag{6.17}$$

where
$$A(x, y) = -\Sigma (1 + m^2)^{-1} [\partial\hat{v}_m/\partial t]_{t=0} \cos my. \tag{6.18}$$

It follows that

$$A - \frac{\partial^2 A}{\partial y^2} = -\Sigma \frac{\partial\hat{v}_m}{\partial t} \Big|_{t=0} \cos my = -\frac{\partial v}{\partial t} \Big|_{t=0} = u_0(x), \tag{6.19}$$

the last equality coming from (6.8) applied at $t = 0$. Since v vanishes on the boundaries, the solution is

$$A = u_0(x) (1 - \cosh y \operatorname{sech} \frac{1}{2}b), \tag{6.20}$$

and so, by (6.17), $\mathcal{U}(x, 0) = u_0(x)$ as required. The proof that $\mathcal{N}(x, 0) = \eta_0(x)$ is similar, following from (6.14) applied at $t = 0$.

The complete solution for the finite channel is now given by (6.1), where \hat{v}_m satisfies (6.3), and by (6.13), (6.14), (6.5) and (6.6). Consider, for instance, the case where the initial conditions are given by (2.6). Then (6.3) becomes

$$\begin{aligned} \partial^2\hat{v}_m/\partial x^2 - \partial^2\hat{v}_m/\partial t^2 - (1 + m^2)\hat{v}_m &= -2\alpha_m \delta(x) \\ &= -2\alpha_m(1 + m^2)^{\frac{1}{2}}\delta((1 + m^2)^{\frac{1}{2}}x). \end{aligned}$$

Comparing with equation (5.14) for the solution $V(x, t)$ for the infinite-width channel, it follows that

$$\hat{v}_m = (1 + m^2)^{-\frac{1}{2}}\alpha_m V((1 + m^2)^{\frac{1}{2}}x, (1 + m^2)^{\frac{1}{2}}t) \tag{6.21}$$

$$= \begin{cases} \alpha_m \int_0^{(t^2-x^2)^{\frac{1}{2}}} (x^2 + s^2)^{-\frac{1}{2}} J_0((1 + m^2)^{\frac{1}{2}}s) s ds & \text{for } |x| < t \\ 0 & \text{for } |x| > t \end{cases} \tag{6.22}$$

by (5.21). Hence, by (6.1),

$$v = \begin{cases} \int_0^{(t^2-x^2)^{\frac{1}{2}}} s(x^2 + s^2)^{-\frac{1}{2}} (\Sigma \alpha_m J_0((1 + m^2)^{\frac{1}{2}}s) \cos my) ds & \text{for } |x| < t, \\ 0 & \text{for } |x| > t. \end{cases} \tag{6.23}$$

Expressions for u^{ev} and η^{ev} follow by substitution of (6.21) in (6.13) and (6.14). Using (5.13), this gives

$$\left. \begin{aligned} u^{ev} &= \Sigma (1+m^2)^{-1} \alpha_m U((1+m^2)^{\frac{1}{2}}x, (1+m^2)^{\frac{1}{2}}t) \cos my \\ &\quad + \mathcal{U}(x, t) \cosh y \operatorname{sech} \frac{1}{2}b, \\ \eta^{ev} &= \Sigma (1+m^2)^{-1} \alpha_m N((1+m^2)^{\frac{1}{2}}x, (1+m^2)^{\frac{1}{2}}t) \cos my \\ &\quad + \mathcal{N}(x, t) \cosh y \operatorname{sech} \frac{1}{2}b, \end{aligned} \right\} \quad (6.24)$$

which shows how the solution for a finite-width channel is related to the two limiting cases. If $b \rightarrow \infty$ (the wide-channel limit), $\operatorname{sech} \frac{1}{2}b \rightarrow 0$ so the terms involving \mathcal{U} and \mathcal{N} do not contribute. In addition, (6.2) shows that $m \rightarrow 0$ so the Fourier series tend to $U(x, t)$ and $N(x, t)$ as required. In the opposite limit, $b \rightarrow 0$, (6.2) shows that $m \rightarrow \infty$ so the Fourier series tend to zero. On the other hand, $\cosh y \operatorname{sech} \frac{1}{2}b \rightarrow 1$ so $u^{ev} \rightarrow \mathcal{U}$ and $\eta^{ev} \rightarrow \mathcal{N}$, i.e. the narrow-channel solutions of §3 are obtained.

7. The solution for large times

A property of the finite-channel solution of special interest is its form for large times. For $t \rightarrow \infty$, §3 gives

$$\mathcal{U} = -1, \quad \mathcal{N} = 0$$

and §5 gives

$$U = 0, \quad N = M(x)$$

with M given by (5.7). Thus from (6.24),

$$\left. \begin{aligned} u^{ev} &= -\cosh y \operatorname{sech} \frac{1}{2}b, \\ \eta^{ev} &= \Sigma (1+m^2)^{-1} \alpha_m M((1+m^2)^{\frac{1}{2}}x) \cos my, \end{aligned} \right\} \quad (7.1)$$

and hence (6.5) and (6.6) give

$$\left. \begin{aligned} u^{od} &= \Sigma m(1+m^2)^{-1} \alpha_m M((1+m^2)^{\frac{1}{2}}x) \sin my, \\ \eta^{od} &= \sinh y \operatorname{sech} \frac{1}{2}b. \end{aligned} \right\} \quad (7.2)$$

Figure 3 shows this solution for two different values of b (0.6 and 4). Consider some of the properties of this solution.

First note that η^{ev} vanishes on the walls, and so η itself takes constant values

$$\eta = \pm \tanh \frac{1}{2}b \quad \text{at} \quad y = \pm \frac{1}{2}b \quad (7.3)$$

as required by the boundary condition (2.5) and the fact that the steady flow is geostrophic [equation (2.2)]. Associated with the difference in level is a steady transport or flux of fluid from the region of high surface level to that of low surface level. This flux is independent of x and given by

$$\int_{-\frac{1}{2}b}^{\frac{1}{2}b} u \, dy = -2 \tanh \frac{1}{2}b, \quad (7.4)$$

which can be deduced from (7.1), or from (7.3) and the integral of (2.2). In the limit $b \rightarrow 0$, the value of η on the walls tends to zero as required, and the average flux across the channel tends to -1 , which is also consistent with (3.6). As b

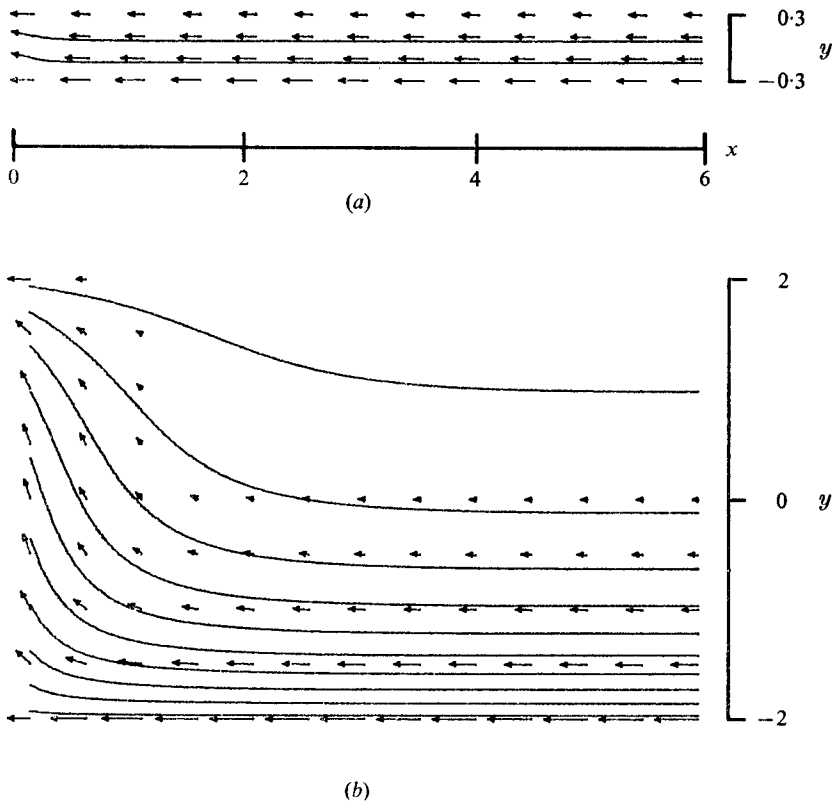


FIGURE 3. The steady solution for a finite channel of width (a) $b = 0.6$ and (b) $b = 4$, the unit of length being the Rossby radius. The region shown is $x > 0$. Contours are of surface elevation at levels 0.1, 0.3, 0.5, etc. The arrows show the current direction and the length of the stem is proportional to the speed.

increases, the difference in surface elevation between the walls increases, with a limiting value of 2. This appears to contradict the wide-channel limiting solution, which is independent of x . The reason is that the wide-channel solution has a boundary-layer structure. Away from the walls, the solution is approximately independent of x and close to the solution of § 5. The differences in level associated with the result (7.3) are associated with boundary layers near the walls.

The boundary-layer structure for large b is most easily determined for large values of $|x|$. Then (7.1) gives

$$\eta^{ev} = \Sigma (1 + m^2)^{-1} \alpha_m \cos my \operatorname{sgn} x \tag{7.5}$$

by (5.7). Using the argument which allowed (6.20) to be deduced from (6.19), the sum in (7.5) can be evaluated to give

$$\eta^{ev} \rightarrow (1 - \cosh y \operatorname{sech} \frac{1}{2}b) \operatorname{sgn} x \quad \text{for } |x| \rightarrow \infty. \tag{7.6}$$

Now using the expression (7.2) for η^{od} , it follows that

$$\eta \rightarrow \left\{ \begin{array}{ll} 1 - e^{-y} \operatorname{sech} \frac{1}{2}b & \text{for } x \rightarrow \infty, \\ -1 + e^y \operatorname{sech} \frac{1}{2}b & \text{for } x \rightarrow -\infty. \end{array} \right\} \tag{7.7}$$

Also, since this steady flow is geostrophic,

$$u \rightarrow \begin{cases} -e^{-y} \operatorname{sech} \frac{1}{2}b & \text{for } x \rightarrow \infty, \\ -e^y \operatorname{sech} \frac{1}{2}b & \text{for } x \rightarrow -\infty. \end{cases} \quad (7.8)$$

In particular, when b is large, the flow takes place in currents confined to the boundaries, with the associated change in surface level also near the boundaries. These boundary layers have width equal to the Rossby radius with the current directed towards the region where the surface elevation was initially low. The 'upstream' boundary current ($x \rightarrow \infty$) is against the left-hand wall $y = -\frac{1}{2}b$. The surface level increases from the value -1 on this wall to the undisturbed value $+1$ away from the wall. On the other hand, the 'downstream' boundary current ($x \rightarrow -\infty$) is against the right-hand wall. The surface elevation still increases towards the right, having the undisturbed value -1 in the interior region and a value of $+1$ on the wall.

Some properties of the solution at the centre-line $x = 0$ can also be deduced. Here $M = 0$ and so (7.1) and (7.2) give

$$\left. \begin{aligned} u &= -\cosh y \operatorname{sech} \frac{1}{2}b, \\ \eta &= \sinh y \operatorname{sech} \frac{1}{2}b. \end{aligned} \right\} \quad (7.9)$$

When b is large, these expressions show that the flux in the x direction is equally divided between boundary layers on the two walls, and the changes in the surface elevation on the line $x = 0$ are also divided between these two boundary layers. The v velocity component also has a boundary-layer structure and is given by

$$v(0, y) = \Sigma (1 + m^2)^{-\frac{1}{2}} \alpha_m \cos my. \quad (7.10)$$

This expression tends to unity away from the walls as $b \rightarrow \infty$ and falls to zero in boundary layers near the walls. Thus the striking asymmetric pattern of figure 3(b) is obtained. The current coming from $x = \infty$ hugs the left-hand wall until it reaches the position of the initial discontinuity in level. There it crosses the channel and then follows the right-hand wall towards $x = -\infty$. The contours of surface height are also streamlines, with high values in the region where x and y are positive and low values where x and y are negative.

Energy changes between the initial and final states can be calculated from the above expressions. Using the definitions of § 4, the following results are obtained for large L , terms exponentially small as $L \rightarrow \infty$ being neglected:

$$P_i - P_f = 2L \tanh \frac{1}{2}b + p(b), \quad (7.11)$$

where

$$p(b) = \frac{3}{4}b \Sigma (1 + m^2)^{-\frac{3}{2}} \alpha_m^2 = \frac{3}{4}b \sum_{n=0}^{\infty} \left(\frac{4}{(2n+1)\pi} \right)^2 \left[1 + \left(\frac{(2n+1)\pi}{b} \right)^2 \right]^{-\frac{3}{2}}, \quad (7.12)$$

$$P_i - P_f - K_f = r(b), \quad (7.13)$$

where

$$r(b) = \frac{1}{2}b \Sigma (1 + m^2)^{-\frac{3}{2}} \alpha_m^2 = \frac{1}{2}b \sum_{n=0}^{\infty} \left(\frac{4}{(2n+1)\pi} \right)^2 \left[1 + \left(\frac{(2n+1)\pi}{b} \right)^2 \right]^{-\frac{3}{2}}. \quad (7.14)$$

The functions $p(b)$ and $r(b)$ are shown in figure 4.

For small b , $p(b) \simeq 0.00397b^6$ and $r(b) \simeq 0.0263b^4$. Thus $P_i - P_f \simeq bL$ and

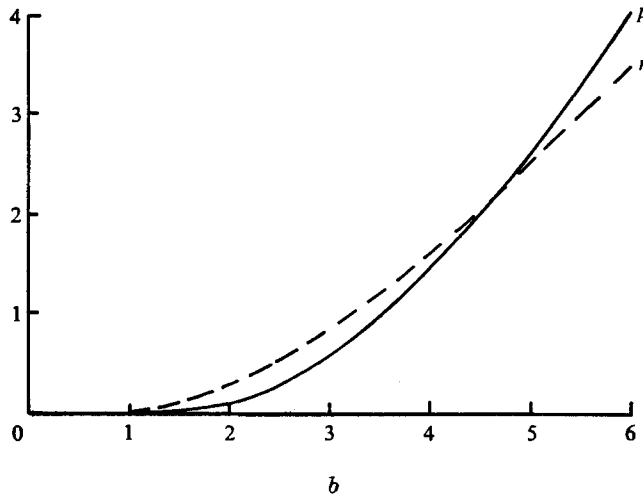


FIGURE 4. The functions $p(b)$ and $r(b)$ associated with the energy changes in a region of length $2L$ of a channel of width b , the unit of length being the Rossby radius and the unit of energy being, for a homogeneous fluid, $\rho H(g\eta_0/f)^2$, where ρ is the fluid density, g the gravitational acceleration, H the undisturbed depth, η the initial surface elevation and f the Coriolis parameter. For large L , the change in potential energy is $2L \tanh \frac{1}{2}b + p(b)$ and the energy lost by radiation is $r(b)$.

$K_f \simeq bL$ in conformity with the narrow-channel result (4.6). For large b , $p \simeq \frac{3}{2}b - \gamma$ and $r \simeq b - \frac{1}{2}\gamma$, where $\gamma = 5.093$. Thus, for $b \rightarrow \infty$ and L fixed, the change in potential energy per unit width is

$$b^{-1}(P_i - P_f) = \frac{3}{2} + O(b^{-1})$$

and the change in kinetic energy per unit width is

$$b^{-1}K_f = 1 + O(b^{-1}).$$

Again these are consistent with the results (5.8) and (5.9) for the wide-channel limit. Note, however, that there is a contribution proportional to L which is due to the adjustment in the boundary layers adjacent to the walls.

8. Development of the solution with time

The changes in the solution with time are illustrated for two values of b in figure 5. The plots for the smaller value of b (0.6) show features like that of the narrow-channel limit while the larger value of b (4) is chosen such that the structure for large values of b is evident. In either case, a distinctive feature is the front which moves out from the initial discontinuity. Since both the Kelvin waves and the short Poincaré waves move with unit speed, so does the front. The discontinuities at the front can be calculated from the expressions in §6. For instance, $u = 0$ ahead of the front, but just behind the front $u = -1$ for all y by (6.24) and (6.5). Similarly $\eta = 1$ ahead of the front, but $\eta = 0$ just behind the front. On the other hand, v is zero both ahead of and just behind the front.

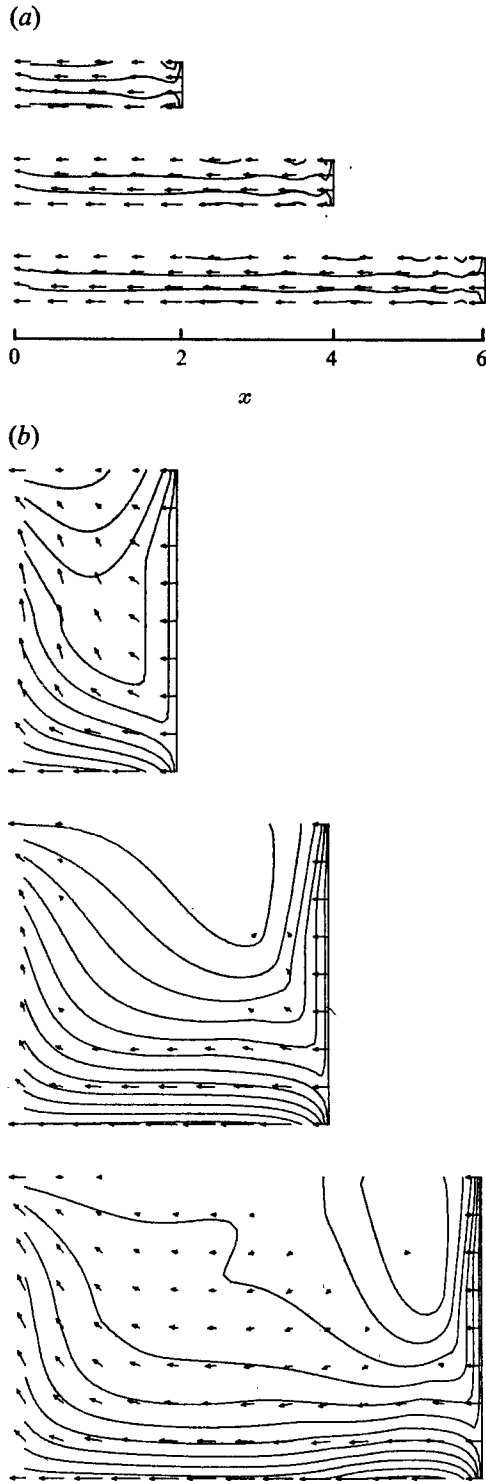


FIGURE 5. The development of the flow with time for a channel of width (a) $b = 0.6$ and (b) $b = 4$. Only the region $x > 0$ is shown and solutions are shown at times $t = 2, 4$ and 6 (the time unit is f^{-1} and the distance unit is the Rossby radius). Contours are of surface elevation at levels $0.1, 0.3, 0.5$, etc. and the arrows indicate the direction and magnitude of the velocity.

The large discontinuities in η and u at the front make this quite a prominent feature of the solutions. In the narrow-channel case, the changes in η and u persist, so that after a long time the values are close to those just behind the front. In the wide channel, on the other hand, the changes in η and u only persist on the side of the channel where the Kelvin wave occurs. In the remainder of the channel, η and u 'recover' after the front has passed and their final values are close to their original ones, except for the region close to the initial discontinuity. The 'wake' of the waves behind the front can be seen even in the case where b is 0.6. There seems little point in saying more, because the figures speak for themselves.

9. Conclusions

One motivation for studying this problem was the idea that it would nicely illustrate the behaviour of a rotating stratified fluid in the presence of boundaries. This it does rather well, showing (a) how boundaries close together can suppress the effects of rotation, (b) the nature of the adjustment well away from boundaries, with the fundamental length scale being the Rossby radius, and (c) the development of a 'coastal jet' on the boundary, this being set up by the passage of a Kelvin wave.

Another motivation was concerned with flow of dense water out of a semi-enclosed basin like the Norwegian Sea. The outflow towards the exit from the basin could be carried by currents concentrated on either boundary, or be subdivided between two such boundary currents. How is this subdivision of flux determined? The problem discussed in this paper suggests that if the flow were established by removing an obstruction from the exit, there being an initial discontinuity in the upper surface of the dense water at the exit, the efflux from the basin would come entirely from a boundary current with, in the Northern Hemisphere, the coast on its left. Thus consideration of time development may help to resolve the question of what steady solutions are possible as a final state.

The solution considered applies to any mode in a channel of uniform depth, and for general initial conditions the complete solution can be found as a sum of normal modes. The method cannot be applied when the depth varies. However, the solution would be expected to have the same character in a real ocean basin since long trapped waves on sloping boundaries also propagate in one direction only (Gill & Clarke 1974).

The solutions presented are restricted in validity by the neglect of friction and of nonlinear effects. The latter effects (as a referee kindly pointed out) are particularly interesting, and imply a two-stage adjustment process. The first stage is the adjustment to a geostrophic equilibrium studied in this paper, the speed of adjustment being determined by the propagation velocity $(gH)^{\frac{1}{2}}$. In a weakly nonlinear system, however, there will be a second stage with slower adjustment determined by the particle velocity \mathbf{u} . The simplest case is that of a homogeneous fluid, where potential vorticity is conserved by each vertical line of fluid particles. The linear solution assumes infinitesimal velocities, so the potential vorticity at each point is fixed at its initial value. For finite velocities, however, figure 3

shows that potential vorticity initially associated with the region $x > 0$ will be advected into the region $x < 0$, thus initiating the second stage in the adjustment process. Calculation of details of this adjustment would make an interesting study.

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